An Efficient Polynomial Approximation to the Normal Distribution Function and Its Inverse Function

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Abstract

We propose approximations to the normal distribution function and to its inverse function using single polynomials in each case. The absolute error of these approximations is significantly less than those of other approximations available in the literature. We compare all the polynomial approximations empirically by calculating their respective percentage absolute relative errors.

Keywords: Normal distribution, Probabilistic polynomial approximation operator

1. Introduction

The problem of approximation arises in many areas of science and engineering in which numerical analysis and computing are involved. The modern history of the subject may be said to have begun in 1885 with Weierstrass's celebrated approximation theorem on the approximation of continuous functions by polynomials. Later on, Bernstein gave a constructive proof of Weierstrass's Theorem by furnishing explicitly, for every $f \in C[0,1]$, a sequence of polynomials (the Bernstein polynomials) that converge to f

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{x} x^k (1-x)^{n-k} . f\left(\frac{k}{n}\right) \tag{1.1}$$

Polynomial functions are, of course, extremely well-behaved. Thus an approximation to the normal distribution function which employs only a single polynomial is likely to be more efficient than existing approximations and easy to calculate.

Let X be a standard normal random variable and let F be the distribution function of X. We aim to construct single polynomial approximations to both F and to the inverse function F^{-1} of F.

2. The Probabilistic Polynomial Approximation of Sahai (2004)

For the standard normal distribution, we know that

$$
\begin{aligned}
F(3.0) &= 0.5 + \int_0^{3.0} \frac{`1}{\sqrt{2\pi}} \exp(-x^2/2).dx \simeq 0.9987 \\
&= 0.5 + \int_0^{1} \frac{3}{\sqrt{2\pi}} \exp(-4.5x^2)\, dx
\end{aligned}
\tag{2.1}
$$

This places the computation of $F(3.0)$ firmly within the $C[0, 1]$ framework and enables us to use the 'quadrature-polynomial formula using the probabilistic approach' of Sahai (2004) which we briefly describe.

Suppose that we are interested in integrating a continuous function over the interval $[0, 1]$. We divide $[0, 1]$ into n equal intervals. Let $x_i = i/n$ for $i = 0, 1, ..., n$. Consider a point $x \in [0, 1]$. Then, if X is a random point in $[0, 1]$, $P(X \le x) = x$ and $P(X > x) = 1 - x$. Thus, of n randomly chosen points, the expected number of points that are less than or equal to x is nx and the expected number greater than x is $n(1 - x)$.

Now, to devise the weight function $A_k(x)$ associated with the node x_k, we simply place it in the same shoes as x. We know that, using $(n + 1)$ equidistant nodes, for any node x_k, there are k nodes to the left of the node x_k and $(n - k)$ nodes to the right of x_k.Consequently, in this 'probabilistic' setup, the probability that the node x_k is chosen is

$$
\binom{n}{k}\binom{n(1-x)}{n-k}/\binom{n}{n} = \binom{n}{k}\binom{n(1-x)}{n-k} = A_k(x) \tag{2.2}
$$

This equation may be expressed in terms of the Gamma function in order to accommodate any real value of $x \in [0, 1]$.

Therefore, Sahai(2004)'s 'probabilistic polynomial approximation' for the distribution function $F(x)$ is simply

$$
F(x) \simeq 0.5 + \int_0^x \sum_{k=0}^{n} A_k(x) f(x_k) dx \tag{2.3}
$$

where

$$
f(x_k) = 3(\frac{1}{\sqrt{2\pi}}) \exp(-4.5x_k^2) \tag{2.4}
$$

The last integral in Eq. 2.1 has no closed form expression. Most statistical books give the values of this integral in normal tables. These tables may also be used to find the value of x when $\Phi(x)$ is known. Several authors give approximations using polynomials (Chokri, (2003); Johnson (1994); Bailey (1981); Polya (1945)). These approximations give quite high accuracy but require significant amounts of computation and have a maximum absolute error of more than 0.003. Only the Polya approximation

$$
F(x) = 0.5[1 + \sqrt{(1 - \exp(\frac{-2}{\pi}x^2))}] \tag{2.5}
$$

has one term to calculate. The others require more than one term. They are reviewed in Johnson et al (1994) and are as follows:

$$
(1)\ F_1(x) = 1 - 0.5(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5)^{-16}, \tag{2.6}
$$

in which $a_0 = 0.9999998582$, $a_1 = 0.487385796$, $a_2 = 0.02109811045$, $a_3 = 0.003372948927$, $a_4 = 0.00005172897742$ and $a_5 = 0.0000856957942$.

$$
(2)\ F_2(x) = \exp(2y)/(1 + \exp(2y)),\ \text{where } y = 0.7988x(1 + 0.04417x^2). \tag{2.7}
$$

$$
(3)\ F_3(x) = 1 - 0.5 \exp[-(83x + 351)x + 562)/(703/x + 165)] \tag{2.8}
$$

$$
(4)\ F_4(x) = 0.5[1 + \sqrt{(1 - \exp(-\sqrt{\frac{\pi}{8}}x^2))}] \tag{2.9}
$$

The last approximation (2.9) was proposed by Aludaat and Alodat (2008) as an improvement on that of Polya's in (2.5). The others require substantial computation, since their inverse functions are quite intricate. Using our probabilistic approximation with $n = 8$, we get the 8th degree polynomial:

$$\sum_{k=0}^{n} A_k(x) f(x) = 1.196826841 - 0.0144665656x - 5.0871271x^2 + 2.3574816544x^3 + 1.32473472x^4 - 18.16066369x^5 - 5.33531361x^6 + 12.93269242x^7 - 4.485957222x^8 \tag{2.10}$$

Hence, using (2.3), we get the following ninth-degree polynomial approximation to the distribution function of the standard normal distribution

$$F5(x) \approx 0.5 + \int_0^x \sum_{k=0}^{n} A_k(x) f(x_k) dx \tag{2.11}$$

$$= 0.5 + 1.196826841x - 0.00723328282x^2 - 1.1695709047x^3 - 0.5893704135x^4 + 0.264946944x^5 - 3.026777282x^6 - 0.7621876586x^7 + 1.616586552x^8 - 0.4984396913x^9 \tag{2.12}$$

Now, we consider the approximation of the inverse function $F^{-1}(p)$, since $F(x) = p \Leftrightarrow F^{-1}(p) = x$, where $0 \leq p \leq 1$. This will have many applications in practical situations. One such application will be in generating random x-values for standard normal variates.

The probability p may be generated using a random number generator with the uniform distribution $U[0, 1]$. If we have generated $p_1, p_2, ..., p_n$ then the inverse distribution function may be used to generate the normal variates $(x_\alpha; \alpha = 1, 2, ..., n)$. We now consider approximations to $F^{-1}(p)$. As $F^{-1}(x)$ in (2.6) would have infinite terms, it could not be expressed in a closed form via a finite degree polynomial. In the abscence of a closed form it would be very tedious to generate a good approximation to the inverse function. Hence we consider only the approximations to the inverse functions given above. These are:

$F^{-1}[2](p)$ = Real root between 0 and 2 of the equation

$$0.7988x(1 + 0.04417x^2) = [\log(p) - \log(1 - p)]/2 \tag{2.13}$$

$F^{-1}[3](p)$ =Real Root between 0 and 2 of the equation

$$(83x + 351)x + 562 + ((703/x) + 165)(\log(2 - 2p)) = 0 \tag{2.14}$$

$$F^{-1}[4](p) = \sqrt{\frac{-\log(1 - (2p - 1)^2)}{\sqrt{\pi/8}}} \tag{2.15}$$

and

$F^{-1}[5](p)$ =Real Root between 0 and 2 of the equation

$$F5(x) - 0.5 = p \tag{2.16}$$

as in (2.12).

2.1 A Numerical Comparison of the Approximations to F(x) and $F^{-1}(x)$

In this section we compare the exact value of $F(x)$ with its approximate ones. We make this comparison for the values $x = 0.1, 0.3, 0.6, 1.0$ and 2.0. These values are tabulated in Table A1 given in the Appendix. The following table, Table A2, gives the values of the Absolute Percentage Relative Error (APRE) for each of the various approximating functions $F(.)(x)$. We calculate $APRE[F(J)(x)]$, where

$$APRE[F(J)(x)] = \frac{|F(J)(x) - F(x)|}{F(x)} 100\% \tag{3.1}$$

The most favourable APRE value has been highlighted.

It is quite evident that our proposed approximation $F5$ is doing well and is consistently better than that of the Aludaat and Alodat (2008) approximation $F4$.

Similarly, we compare the exact value of $F^{-1}(p)$ with its approximated ones. We compare the numerical approximations with the exact values of $F^{-1}(p)$ at the values $p = 0.5539828, 0.617911, 0.725747, 0.841345, 0.933193$ and 0.977250.These are given in Table A3 in the Appendix.

The following table A4 displays the values of the APRE for various approximations to $F^{-1}(p)$.Once again, the most favourable value has been highlighted. Our approximation $F^{-1}[5](p)$ does quite well and is consistently better than that of Aludaat and Alodat (2008).

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APPENDIX.

Table A.1. Values of Various Approximating Functions F (o) (x) & Actual Value of Normal Feller Function F (x)

x-values → Apxg. Fns. ↓	0.1	0.3	0.6	1.0	1.5	2.0
F (1) (x)	0.538972	0.615312	0.719751	0.830390	0.919689	0.966501
F (2) (x)	0.539873	0.618028	0.725877	0.841331	0.933053	0.977240
F (3) (x)	0.539872	0.617933	0.725693	0.841280	0.933172	0.977250
F (4) (x)	0.539519	0.617088	0.724700	0.841184	0.934699	0.979181
F (5) (x)	0.539823	0.617895	0.725733	0.841330	0.933179	0.977234
F (x)-Values:	0.539828	0.617911	0.725747	0.841345	0.933193	0.977250

Table A.2. Values of Abs. Per. Rel. Error [APRE] For Various Approximating Functions F (o) (x)

x-values → Apxg. Fns. ↓	0.1	0.3	0.6	1.0	1.5	2.0
APREF (1) (x)	0.158569	0.420611	0.826183	1.302082	1.447075	1.099923
APREF (2) (x)	0.008336	0.018935	0.017913	0.001664	0.015002	0.001023
APREF (3) (x)	0.008151	0.003560	0.007440	0.007726	0.002250	0.000000
APREF (4) (x)	0.057240	0.133191	0.144265	0.019136	0.161381	0.197595
APREF (5) (x)	0.000926	0.002589	0.001929	0.001783	0.001500	0.001637

Table A.3. Values of Approximating Inverse Functions F-1(o) (p) & Actual Value of The Inverse Function F-1(p)

p-values → Apxg. Fns. ↓	0.539828	0.617911	0.725747	0.841345	0.933193	0.977250
F-1 (2) (p)	0.099887	0.299694	0.599611	1.000057	1.501082	2.000184
F-1 (3) (p)	0.099889	0.299943	0.600163	1.000269	1.500158	2.000006
F-1 (4) (p)	0.100785	0.302171	0.603140	1.000658	1.486901	1.965099
F-1 (5) (p)	0.100013	0.300041	0.600043	1.000061	1.500110	2.000288
F-1(p)-Values:	0.100000	0.299999	0.600000	1.000001	1.500002	2.000002

Table A.4. Values of Abs. Per. Rel. Error [APRE] For Various Approximating Functions F (o) (x)

p-values → Apxg. Fns. ↓	0.539828	0.617911	0.725747	0.841345	0.933193	0.977250
APREF-1 (2) (p)	0.113000	0.101667	0.064833	0.005600	0.072000	0.009100
APREF-1 (3) (p)	0.111000	0.018667	0.027167	0.026800	0.010400	0.000200
APREF-1 (4) (p)	0.785000	0.724002	0.523333	0.065700	0.873399	1.745148
APREF-1 (5) (p)	0.013000	0.014000	0.007167	0.006000	0.007200	0.014300
F-1(p)-Values:	0.100000	0.299999	0.600000	1.000001	1.500002	2.000002