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A Sigmoid Approximation of the Standard Normal Integral

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ABSTRACT

This paper presents a simple sigmoid function for approximation of cumulative standard normal probabilities. The approximation has an error of ± 0.000043 for $-8 \leq Z \leq 8$.

1. INTRODUCTION

Most probability and statistics books, e.g., [1, 2], present the normal density function with the standard normal transformation and give a tabulation of cumulative standard normal probabilities. Reference is commonly made to the fact that the probabilities are obtained by integrating the normal density function. However, because the integration of the normal density function cannot be done by elementary methods, various approximations are used to determine cumulative standard normal probabilities.

The goal here is to develop one simple, yet powerful, function that gives good approximations of the cumulative standard normal probabilities for most practical applications. This article assumes that the reader is familiar with the functional form of the normal density, as well as the use of cumulative normal distribution tables found in most probability and statistics books.

2. OTHER APPROXIMATIONS

Several approximations were found in the literature. Among these are convergent series expansions (presented in many books, including [3, 4]). Selected other approximations (in chronological order) have been proposed, e.g., by Hastings [5] (presented also in [6]), Burr [7], Hill and Joyce [8],

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Hoyt [9], Strecok [10], Cody [11], Milton and Hotchkiss [12], Adams [13], Johnson and Kotz [14], Shah [15], and Norton [16].

The above-referenced approximations can be divided into two groups. In one group are the approximations striving for highest accuracy, and in the other group are the approximations attempting simplicity while maintaining a level of practical accuracy. The approximations do not necessarily improve over time, but offer different, and in some cases very simple, formulas for finding cumulative standard probabilities. A brief review of some of the cited approximations in the above two groups follows.

In, e.g., [3, 4], the convergent series expansion of the standard normal integral is presented. The degree of accuracy depends on the number of series terms evaluated. One needs to evaluate 30 terms of the convergent series to cover an interval of $-4.2 \leq Z \leq +4.2$ with an absolute maximum error of 6.9×10^{-8} within the interval. The 30-term approximation deteriorates rapidly outside the above interval. The convergent series expansion for k terms for $Z \leq z$ has the following form:

$$F(Z \leq z) \approx 0.5 + \frac{1}{\sqrt{\pi}} \left[\sum_{i=1}^{k} \pm \frac{1}{(2i-1)(i-1)!} \left(\frac{z}{\sqrt{2}} \right)^{2i-1} \right]. \tag{1}$$

Strecok [10] presented an approximation to the error function ($\mathrm{erf}(z)$) and used the known relationship between $\mathrm{erf}(z)$ and the standard normal distribution to determine cumulative normal probabilities. The approximation is accurate to 16 decimal places for $-8 \leq Z \leq +8$ with $k = 37$, where k is the number of terms evaluated of the $\mathrm{erf}(z)$ approximation (Eq. (2)). The relationship between k, maximum Z-range, and the size of the absolute error when using (Eq. (2)) is, e.g., as follows: $k = 12$, with $-4.2 \leq Z \leq +4.2$, error 4.1×10^{-5}; $k = 20$, with $-6.1 \leq Z \leq +6.1$, error 7.2×10^{-8}.

$$\mathrm{erf}(z) \approx \frac{2}{\pi} \left[\frac{z}{5} + \sum_{i=1}^{k} \frac{e^{-(i/5)^2} \sin(2iz/5)}{i} \right] \tag{2}$$

Cody [11] proposed an approximation of the $\mathrm{erf}(z)$ using rational fractions in three positive Z-range intervals and then converted the $\mathrm{erf}(z)$ values to normal probabilities. The rational fractions contain polynomials up to the eighth degree. The number of polynomial terms to be evaluated depends on the interval of Z and varies between 8 and 16. Accuracy was reported to be approximately 25 decimal places.

Burr [7] proposed two simple polynomial type formulas for the interval $-3.9799 \leq Z \leq +3.9799$ with an absolute maximum error of 4.7×10^{-4}. Burr's approximation is not designed to work outside the above interval.

Hoyt [9] divided the positive Z-range into three intervals and proposed a simple polynomial for each interval achieving an accuracy, in terms of maximum absolute error, of 1.0×10^{-2}. Hoyt's approximation is not designed to work outside $-3 \leq Z \leq +3$ interval. Johnson and Kotz [14] proposed a single exponential function for the positive Z-range with an absolute error of 2×10^{-2}. Shah [15] divided the positive Z-range into three intervals and proposed a second-degree polynomial for one interval, and constants to the others, achieving an accuracy of 5.1×10^{-3}. Shah's approximation is not designed to work outside $-2.6 < Z < +2.6$. Norton [16] divided the positive Z-range into two intervals and proposed exponential function approximations for each interval with an absolute error of 8×10^{-3}.

The above-referenced series expansion approximations [3, 4, 10] and the rational fraction approximations (e.g., [11]) are very accurate and cover a range $-8 \leq Z \leq +8$, if a sufficient number of terms are evaluated. These types of approximations are also computationally demanding and are, therefore, best suited for software implementations.

Most, if not all, of the second group approximations are multiformula approximations and have attempted simplicity, while maintaining acceptable practical accuracy. These approximations appear to suffer either from lack of coverage, in terms of Z-range, or accuracy, or both.

The purpose of this paper is to provide an approximation that is a significant improvement to the second category of approximations. The proposed single formula approximation is simple, has an improved accuracy, and it covers a range $-8 \leq Z \leq +8$.

3. THE SIGMOID APPROXIMATION

The proposed approximation function has the form of a sigmoid function. It can be used to determine approximate cumulative probabilities of the standard normal distribution. The approximation function, for $-8 \leq Z \leq +8$, has the form

$$F(Z \leq z) \approx \frac{1}{1 + e^{-\sqrt{\pi}(\beta_1 z^5 + \beta_2 z^3 + \beta_3 z)}} \tag{3}$$

where $\beta_1 = -0.0004406, \beta_2 = +0.0418198, \beta_3 = +0.9000000$, and $Z = (X - \mu)/\sigma$ is a standard normal random variable with value z.

The parameters β_1, β_2, and β_3 were estimated to seven decimal places. The accuracy of the approximation will improve somewhat if parameter estimates are further refined.

The error of this approximation varies between ± 0.0000431 (or $\pm 4.31 \times 10^{-5}$) for $-8 \leq Z \leq +8$. The error reaches the minimum and maximum values alternating at Z values ± 2.2, ± 1.1, and follows a distinct predictable pattern. The absolute error decreases significantly toward the tails of the normal distribution. For example, for $Z = \pm 3$, ± 4, ± 5, ± 6, ± 7, ± 8 the absolute errors are 1.2×10^{-5}, 1.1×10^{-6}, 8.6×10^{-8}, 2.4×10^{-9}, 6.3×10^{-11}, 1.2×10^{-11}, respectively.

Additional higher-order terms in the polynomial exponent will improve the approximation slightly, but not significantly. The trade-off between the amount of additional work from added higher-order terms to the polynomial, or added corrective terms to the approximation function, and the improvement of accuracy does not appear to be warranted for most practical applications.

4. CONCLUSION

This paper presents a simple approximation (Eq. (3)) for finding cumulative standard normal probabilities. The proposed sigmoid approximation is a single formula for $-8 \leq Z \leq +8$ with an absolute error of 4.31×10^{-5}. The closed functional form can easily be evaluated when neither normal probability tables nor appropriate software are readily available. It can also be easily embedded in other models and formulas, which require inputs from the normal cumulative distribution.

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