An Efficient Polynomial Approximation of the Normal Distribution Function & Its Inverse Function

[2]Winston A. Richards, [1]Robin Antoine, *[1]Ashok Sahai, and [3]M. Raghunadh Acharya
[1]Department of Mathematics & Computer Science; Faculty of Science & Agriculture
St. Augustine Campus; The University of The West Indies.
Trinidad & Tobago; West Indies.
[2]Department of Mathematics and Statistics
The Pennsylvania State University, Harrisburg, USA
[3]Department of Statistics and Computer Science,
Aurora's Post Graduate College; Osmania University,
Hyderabad (Andhra Pradesh); India.

Abstract

In this paper, we propose a single-polynomial approximation to the normal cumulative distribution function, as also that of the inverse of the normal cumulative distribution function too, using this polynomial. Our approximation has significantly less absolute error of approximation relative to other popular approximations available in the literature, including the recent improved approximation achieved by Aludaat K. M. & M. T. Alodat (2008). This paper is motivated by the powerful polynomial approximation operator of Sahai (2004), which uses a probabilistic approach. We compare all the competing approximations empirically, relative to the relevant exact values, via calculating their respective Percentage Absolute Relative Errors.

Keywords: Feller function, normal distribution, probabilistic polynomial-approximation operator.

MSC2010: 32E30.

Corresponding Author's Email: sahai.ashok@gmail.com

1. Introduction

The problem of approximation arises in many areas of science and engineering in which numerical analysis and computing are involved. In 1885, Weierstrass proved his celebrated approximation theorem: if f ε C [a, b]; for every δ > 0; there is a polynomial 'p' such that $|f - p| < \delta$.

This result marked the beginning of mathematicians' interest in 'polynomial approximation of an unknown function using its values generated experimentally or known otherwise at certain chosen 'Knots' of interest in the domain of relevant variable. Later, Russian mathematician S. N. Bernstein proved the Weierstrass' approximation theorem in a manner which was very stimulating and interesting in many ways.

He first noted a simple but important fact that if the Weierstrass' theorem holds for C [0, 1], it also holds for C [a, b] and conversely. Essentially C [0, 1] and C [a, b] are identical, for all practical purposes, as they are linearly isometric as normed spaces and order isomorphic as algebras (rings).

Perhaps the most important contribution in Bernstein's proof of this theorem consisted in the fact that Bernstein actually displayed a sequence of polynomials that approximate a given function f(x) ε C [0, 1]. If f (x) is any bounded function on C [0, 1], the sequence of Bernstein's Polynomials for f (x) is defined by:

$$(B_n(f))(x) = \sum_{k=0}^{k=n} w_{k,n} * f(k/n) \tag{1.1}$$

Where, $w_{k,n}= \binom{n}{k}x^k(1-x)^{n-k}$ are the respective weights for the values 'f (k/n)' of the function at the knots" (k/n)" [k = 0 (1) n].

The most significant fact to be noted, at this stage, is that any polynomial is such a 'nice' continuous function. The polynomials are infinitely integrable and may be differentiated arbitrarily many times till they cease to exist.

Thus, our proposed single-polynomial approximation of the normal distribution function will not only be more efficient than existing approximations but will also be very easy to calculate, even with a pocket calculator.

We proceed to introduce this in what follows.

Let *X* be the standard normal random variable, i.e., a random variable with the following probability density function.

$$f(x) = (1/\sqrt{(2\pi)}) \exp(-x^2/2); \qquad -\infty < x < \infty. \tag{1.2}$$

Hence, the distribution function [F(x) = P (X < x)] for the standard normal random variable X is:

$$F(x) = \int_{\infty}^{x} (1/\sqrt{2\pi}) \exp(-t^2/2)dt \tag{1.3}$$

The aim of this paper is to achieve a single-polynomial approximation for approximating F (x) in (1.3) above, and the inverse distribution function F^{-1} (x)= IF(x) (Say), for the standard normal distribution.

2. The Efficient Probabilistic Polynomial Approximation Operator of Sahai (2004).

We know that for the standard normal distribution, the value of the distribution function F (x) for x=3.0 happens to be:

$$\mathbf{F\ (3.0) = 0.9987 = 0.5 +} \int_0^{3.0} (1/\sqrt{2\pi}).\exp(-x^2/2)dx \qquad \mathbf{(2.1)}$$

Hence, for our proposed polynomial-approximation the target is effectively: ≈

$$\approx \int_0^{3.0} (1/\sqrt{2\pi}).\exp(-x^2/2)dx \equiv \int_0^{1} (3/\sqrt{2\pi}).\exp(-(4.5)*x^2)dx \qquad \mathbf{(2.2)}$$

Now, this conforms to the ambit-interval C [0, 1] of Sahai (2004)'s "computerizable quadrature-polynomial formula using the probabilistic approach". It is desirable, for ease of reference, to detail here the genesis of this 'simple probabilistic polynomial approximator' to be used for our target as in (2.1), above.

As such, the interval of integration happens to be [0, 1], while x_0= 0 and x_n= 1. We consider the equidistant nodes: x_i= (i/n); i= 0, 1, 2. . . n. (2.3)

Now, considering the line [0, 1], let us visualize a randomly sitting point x on it. It is obvious that the probability of a point on this line being less than x (on its left, on the line) is x, whereas the probability of a point on this line being more than x (on its right, on the line) is 1 - x, i.e., P(X < x) = x and P(X > x) = 1 – x. (2.4)

Hence, the expected number of points out of n equidistant-points on the line which are on the left of the point x (or smaller than x) will be nx, and the expected number of points out of n equidistant points on the line which are on the right of the point x (or greater than x) will be (n – n.x) or equivalently n (1 – x).

Now, to devise the weight function 'A_k (x)' associated with the node x_k, we simply place it in the shoes of x. However, we know that according to our choice of the n + 1 nodes in (2.3), for any node x_k there are k nodes on the left of the node x_k, and that there are (n – k) nodes on the right of the node x_k. Consequently, in this probabilistic setup, the probability of our choice of the node x_k is

$$\binom{nx}{k}*\binom{n.(1-x)}{n-k}/\binom{n}{n} \equiv \binom{nx}{k}*\binom{n.(1-x)}{n-k} = A_k\,(x) \qquad \left[\text{As}\binom{n}{n}=1\right] \qquad (2.5)$$

The equation in (2.5) might well be expressed in terms of the well-known Gamma functions for computational purposes to accommodate any real value of x in [0, 1].

Therefore, the 'probabilistic polynomial approximator' for the distribution function F (x), as in (2.2) (resulting from using the aforesaid probabilistic perspective of polynomial approximation) is simply:

$$F\,(x) \approx 0.5 + \int_0^x \sum_{k=0}^{k=n} A_k(x) * f(x_k).dx\text{; wherein } f\,(x_k) = 3.(1/\sqrt{2\pi})\exp(-(4.5)*(x_k)^2 \qquad (2.6)$$

The last integral in (2.2) has no closed form. Most basic statistical books give the values of this integral for different values of *x* in a table called the standard normal table.

From this table we can also find the value of *x* when Φ(*x*) is known. Several authors gave approximations using polynomials (Chokri, 2003; Johnson, 1994; Bailey, 1981; Polya, 1945).

These approximations give quite high accuracy, but computer programs are needed to obtain their values and they have a maximum absolute error of more than .003. Only the Polya's approximation

$F(x) = 0.5*[1+\sqrt{(1- \exp((-2/\pi)*x^2)}]$ (2.7)

has one-term to calculate, while the others need more than one term. They are reviewed in Johnson et al. (1994) as follows:

1. **Let $F1(x) \approx 1 – 0.5*(a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 + a_6 x^5)^{-16}$; wherein, (2.8)**

   $a_1 = 0.9999998582$, $a_2 = 0.487385796$, $a_3 = 0.02109811045$, $a_4 = 0.003372948927$, $a_5 = 0.00005172897742$, and $a_6 = 0.0000856957942$.

2. **Let $F2(x) \approx 1- (2.\pi)^{-1/2} * \exp(- 0.5 x^2 - 0.94 x^{-2})$, [$x \geq 5.5$/ Thus. excluded!] (2.9)**

3. **Let $F2(x) \approx \exp(2*y) / (1 + \exp(2*y))$, $y = 0.7988.x (1 + 0.04417. x^2)$. (2.10)**

4. **Let $F3(x) \approx 1 – 0.5*\exp[- (83*x +351)*x + 562) / (703/x + 165)]$. (2.11)**

5. **Let $F4(x) \approx 0.5*[1 + \sqrt{(1 - \exp(-\sqrt{(\pi/8)}*x^2))}]$ (2.12)**

Whereas the approximation in (2.12) was proposed by Aludaat and Alodat (2008) as an improvement of that by Polya’s in (2.7), all the other approximations need computer programs to calculate, since their inverse functions are quite intricate and implicit.

Now, using our probabilistic polynomial approximation as in (2.6) with n=8 [i.e. '9' knots], we get the 8th.-degree polynomial:

$\sum_{k=0}^{k=n} A_k(x) * f(x_k)$ **=1.196826841-0.0144665656.x–5.0871271.x^2–2.3574816544.x^3+ 1.32473472.x^4– 18.16066369. x^5–5.33531361.x^6+12.93269242.x^7–4.485957222.x^8. (2.13)**

And hence, using (2.6), we get the following single-polynomial approximation to the distribution function of the standard normal, a ninth-degree polynomial:

6. **Let F5 (x) ≈ 0.5 +** $\int_0^x \sum_{k=0}^{k=n} A_k(x) * f(x_k).dx$ **= 0.5 + 1.196826841.x – 0.00723328282.x^2 – 1.695709047.x^3- 0.5893704135.x^4 + 0.264946944.x^5 – 3.026777282.x^6 - 0.7621876586.x^7 + 1.616586552.x^8 – 0.4984396913.x^9. (2.14)**

Now, as mentioned earlier in the introduction, we will take up the approximation of the inverse distribution function" $F^{-1}(p)$ [F (x) = p ≈ $F^{-1}(p)$ =x (where 0 ≤ p ≤ 1)]. This will have many applications in practical situations. One such application will be in generating random x-values for standard Normal variate.

The probability p (0 ≤ p ≤ 1) may be generated using a random-number generator from the Uniform Distribution U [0, 1]. Suppose we would have generated p_1, p_2, p_3, …, p_n that could be used to generate the standard Normal Variates: { x_α; α = 1(1) n}, using the inverse distribution Function for the Standard Normal distribution, namely "$F^{-1}(p_\alpha)$" [α = 1 (1) n]. Therefore we now consider the approximations to $F^{-1}(p)$ in what follows. As F1 (x) in (2.8) would have infinite terms, it could not be expressed in a closed form via a finite degree polynomial.

In a 'Closed form', it would very implicit and tedious to generate a good approximation to the inverse function, $F^{-1}[1]$ (p). Hence we consider only the approximations to the inverse functions, say $F^{-1}[I]$ (p); I = 2 (1) 5 in (2.10), (2.11), (2.12) and (2.14), as follows.

$F^{-1}[2]$ (p) = Real Root [Between 0 to 2] of the equation: ~

$$0.7988.\ x.\ (1+0.04417.x^2) = \{\log (p) - \log (1-p)\}/2. \qquad (2.15)$$

$F^{-1}[3]$ (p) = Real Root [Between 0 to 2] of the equation: ~

$$(83.x + 351).x + 562 + ((703/x) + 165).\ (\log (2-2.p)) = 0. \qquad (2.16)$$

$$F^{-1}[4]\ (p) = \sqrt{[- \log (1- (2.p-1)^2)/ \sqrt{(\pi/8)}]} \qquad (2.17)$$

And; $F^{-1}[5]$ (p) = Real Root [Between 0 to 2] of the equation: ~

$$F5\ (x) - 0.5 = p. \qquad [F5\ (x), \text{ as per } (2.14)] \qquad (2.18)$$

3. A Numerical Comparison of the Approximations to F (x) and F^{-1}(p).

In this section, we compare the exact value of F (*x*) with its approximated ones, namely F1 (x), F2 (x), F3 (x), F4 (x), and F5 (x) [As per their expressions in equations (2.8), (2.10), (2. 11), (2.12), and (2.14), in the preceding section. The comparison is afforded per their numerical values so calculated vis-à-vis the exact value of F (*x*), for each of the illustrative example-values of 'x' (= 0.1, 0.3, 0.6, 1.0, 1.5, and 2.0), respectively.

These values are tabulated in the 'Table A.1' given in the APPENDIX. The following 'Table A.2' displays the values of Abs. Per. Rel. Error [APRE] For Various Approximating Functions F (•) (x).

Wherein; APRE [F (J) (x)] = $\frac{|F(J)(x) - F(x)| * 100\%}{F(x)}$**; J = 1 (1) 5. The most favourable 'approximation F (•) (x)'s value/ APRE value has been highlighted.**

It is quite evident that our proposed approximation F5 (x) is doing rather well, and is consistently better than that by Aludaat and Alodat (2008) approximation F4 (x)!

Similarly, we compare the exact value of F^{-1}(p) with its approximated ones, namely F^{-1}[2] (p), F^{-1}[3] (p), F^{-1}[4] (p), and F^{-1}[5] (p) [As per their expressions in equations (2.15), (2. 16), (2.17), and (2.18), in the preceding section. The comparison is afforded per their numerical values so calculated vis-à-vis the exact value of F^{-1} (p), for each of the illustrative example-values of 'p' (=0.539828, 0.617911, 0.725747, 0.841345, 0.933193, and 0.977250), respectively, tabulated in 'Table A.3' in the APPENDIX.

The following 'Table A.4' displays the values of Abs. Per. Rel. Error [APRE] for various approximating functions F^{-1} (•) (p). Wherein; APRE [F^{-1} (J) (p)] = $\frac{|F^{-1}(J)(p) - F^{-1}(p)| * 100\%}{F^{-1}(p)|}$**; J = 1 (1) 5. The most favourable 'approximation F^{-1} (•) (p)'s value/ APRE value has been highlighted; making it evident that our proposed approximation "F^{-1}[5] (p)" is doing rather well, and is consistently better than that by Aludaat and Alodat (2008) approximation F4 (x)!**

References

[1] Aludaat, K.M. and Alodat, M.T. (2008). 'A note on approximating the normal distribution functions'. Applied Mathematical Sciences; 2, 9,425-29.

[2] Bailey, B. J. R. (1981). 'Alternatives to Hasting's approximation to the inverse of the Normal cumulative distribution Function'. Applied Statistics, 30, (3) 275-276.

[3] Chokri (2003). 'A short note on the numerical approximation of the standard normal cumulative distribution and its Inverse'. *Real 03-T-7, online manuscript*.

[4] Johnson, N. I., Kotz, S. and Balakrishnan, N. (1994); '*Continuous univariate distribution's*. John Wiley & Sons.

[5] LeBoeuf, C., Guegand, J., Roque, J. L. and Landry, P. (1987) *Probabilites* Ellipses.

[6] Polya, G. (1945) Remarks on computing the probability integral in one and two dimensions. *Proceeding of the first Berkele Symposium on mathematical statistics and probability*, 63-78.

[7] Renyi, A. (1970). *Probability theory*. North –Holland.

[8] Sahai, A., Jaju, R.P., and Mashwama, P.M. (2004) 'A new computerizable quadrature formula using probabilistic Approach'. Applied Mathematics and Computation. 158; 217-24.

APPENDIX.

Table A.1: Values of Various Approximating Functions F (•) (x) & Actual Value of Normal Feller Function F (x).

x-values → Apxg. Fns. ↓	0.1	0.3	0.6	1.0	1.5	2.0
F (1) (x)	0.538972	0.615312	0.719751	0.830390	0.919689	0.966501
F (2) (x)	0.539873	0.618028	0.725877	0.841331	0.933053	0.977240
F (3) (x)	0.539872	0.617933	0.725693	0.841280	0.933172	0.977250
F (4) (x)	0.539519	0.617088	0.724700	0.841184	0.934699	0.979181
F (5) (x)	0.539823	0.617895	0.725733	0.841330	0.933179	0.977234
F (x)-Values:	**0.539828**	**0.617911**	**0.725747**	**0.841345**	**0.933193**	**0.977250**

Table A.2: Values of Abs. Per. Rel. Error [APRE] For Various Approximating Functions F (•) (x).

x-values → Apxg. Fns. ↓	0.1	0.3	0.6	1.0	1.5	2.0
APREF (1) (x)	0.158569	0.420611	0.826183	1.302082	1.447075	1.099923
APREF (2) (x)	0.008336	0.018935	0.017913	0.001664	0.015002	0.001023
APREF (3) (x)	0.008151	0.003560	0.007440	0.007726	0.002250	0.000000
APREF (4) (x)	0.057240	0.133191	0.144265	0.019136	0.161381	0.197595
APREF (5) (x)	0.000926	0.002589	0.001929	0.001783	0.001500	0.001637

Table A.3: Values of Approximating Inverse Functions F^{-1}(•) (p) & Actual Value of The Inverse Function F^{-1}(p).

p-values → Apxg. Fns. ↓	0.539828	0.617911	0.725747	0.841345	0.933193	0.977250
F^{-1} (2) (p)	0.099887	0.299694	0.599611	1.000057	1.501082	2.000184
F^{-1} (3) (p)	0.099889	0.299943	0.600163	1.000269	1.500158	2.000006
F^{-1} (4) (p)	0.100785	0.302171	0.603140	1.000658	1.486901	1.965099
F^{-1} (5) (p)	0.100013	0.300041	0.600043	1.000061	1.500110	2.000288
F^{-1}(p)-Values:	**0.100000**	**0.299999**	**0.600000**	**1.000001**	**1.500002**	**2.000002**

Table A.4: Values of Abs. Per. Rel. Error [APRE] For Various Approximating Functions F (•) (x)

p-values → Apxg. Fns. ↓	**0.539828**	**0.617911**	**0.725747**	**0.841345**	**0.933193**	**0.977250**
APREF^{-1} (2) (p)	0.113000	0.101667	0.064833	0.005600	0.072000	0.009100
APREF^{-1} (3) (p)	0.111000	0.018667	0.027167	0.026800	0.010400	0.000200
APREF^{-1} (4) (p)	0.785000	0.724002	0.523333	0.065700	0.873399	1.745148
APREF^{-1} (5) (p)	0.013000	0.014000	0.007167	0.006000	0.007200	0.014300
F^{-1}(p)-Values:	**0.100000**	**0.299999**	**0.600000**	**1.000001**	**1.500002**	**2.000002**