# A UNIFORM APPROXIMATION TO THE RIGHT NORMAL TAIL INTEGRAL

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Abstract. This paper presents two simple formulas for approximation of the standard normal right tail probabilities, and indicates the method of computing approximations of higher order. One of the approximations relies on two simple numerical constants, has absolute error of 0.00071 and relative error 0.023; the other uses four numerical constants, has absolute error of 0.000019, the relative error is .005 and gives at least two significant digits for probabilities over the entire range $0 \leq z < \infty$.

## 1. Introduction

There are two groups of approximations to standard normal tail probabilities

$$\Pr(Z > z) = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx,\ z \geq 0,$$

found in the literature. One group consists of approximations that have, at least in theory, arbitrarily high precision. We will call those "numerical algorithms". Another group consists of approximations that offer no simple means of increasing their precision – these we will call "ad-hoc approximations". Numerical algorithms often require massive computations, while the ad hoc approximations are often based on short formulas, and few carefully selected numeric constants; so in broad outline our classification agrees with Waissi and Rossin [1]. However, there are exceptions in both groups: some numerical algorithms combine precision with "pocket calculator" simplicity, see Section 2.1.1; on the other hand, some hard to improve and very accurate ad hoc algorithms require a moderately large number of steps, see Section 2.4.1.

A notable feature of most of the approximations in both groups is that they are designed to work in a predefined range of the values of $z$, deteriorate rapidly outside of this range, and the relative errors of approximations to small tail probabilities are unbounded. The sole exception to this rule are two ad-hoc approximations of Hart [2] and [3], quoted here as (10) and (11) respectively, which have uniformly bounded relative errors. (Hart's formulas have been overlooked, and many subsequent publications propose approximate formulas that are more complicated, less accurate, and have narrower range.) As far as we know, there is no single numerical procedure with uniform relative errors over the entire range $z \geq 0$. Of course, one can easily combine a numerical algorithm designed for small $z$ with a numerical algorithm designed for large $z$, so the distinction is a bit academic.

Our goal here is to develop an approximation that has relative errors uniformly bounded across all values of $z$, small absolute errors, is simple enough to produce a "calculator approximation" yet it comes from the well defined procedure that offers the potential for increasing the precision.

## 2. Overview of approximations

2.1. **Series Expansions.** Expansions (1), (2), can be found in Laplace's book [4, p. 103]. The Taylor's polynomial approximation is

$$\Pr(Z > z) \approx \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{N} (-1)^k \frac{z^{2k+1}}{k! 2^k (2k+1)} \tag{1}$$

see e.g. [5, 26.2.10]. The size of the polynomial $N$ can be selected dynamically for each $z$; in theory arbitrary accuracy can be achieved but in practice the series is difficult to use for $z > 6$, even when evaluated at integer values of $z$ with "arbitrary precision" of a symbolic program like Maple, or Mathematica.

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An expansion that fares a bit better is

$$\Pr(Z > z) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!!} \tag{2}$$

see e.g. [5, 26.2.11] With careful programming, and using Maple, the latter expansion is quite effective. For example, expansion truncated at 250 terms gives $\Pr(Z > 12) \approx 1.77648211207767899769 6 \times 10^{-32}$ with 22 significant digits. The "exact values" in our Table 1 and Table 2 were produced in this manner.

McConnell [6] credits Texas Instrument Instruction booklet with the approximation $\Pr(Z > z) \approx \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \sum_{j=1}^{5} \frac{b_j}{(1+pz)^j}$, where $p = .2316419$, $b_1 = .31938153$ $b_2 = -.356563782$ $b_3 = 1.781477937$ $b_4 = -1.821255978$ $b_5 = 1.330274428$. This 6-constant expression has absolute errors of $7.5 \times 10^{-8}$

2.1.1. *Sigmoid approximations.* Revfeim [7] points out a class of sigmoid approximations

$$\Pr(Z > z) = 1/\left(1 + \exp(\sum_{k=0}^{\infty} a_k z^{2k+1})\right) \tag{3}$$

derived from the Taylor expansion for $\ln \frac{P(Z<z)}{\Pr(Z>z)}$. Such approximations are well suited for use within symbolic programs that can compute Taylor expansions of arbitrary order, and perform well as invertible "pocket calculator" approximations. The precision of all sigmoid approximation is sensitive to minute changes in coefficients. For our tests we used the 13-th degree Taylor's expansion computed with Maple.

$$\begin{gathered} \textstyle\sum_{k=0}^{\infty} a_k z^{2k+1} = \\ 2\sqrt{2/\pi} z + .072671205 z^3 - .000073961474 z^5 - .00010432 z^7 + \\ .0000057026697 z^9 + .00000015418240 z^{11} - .000000037062996 z^{13} \ldots \end{gathered} \tag{4}$$

This sum was truncated to one term in [8]; and to two terms in [9], resulting in "calculator approximations" of good precision, see Table 3. Several authors modified the coefficients to achieve better precision. Page [10] gives two sigmoid approximation using

$$a_1 z + a_2 z^3 = 1.5976 z + .070565992 z^3 \tag{5}$$

and

$$a_1 z + a_2 z^3 = 2\sqrt{2/\pi} z + .07????z^3 \tag{6}$$

in (3). Waissi Rossin [1] use (3) with

$$a_1 z + a_2 z^3 + a_3 z^5 = 1.595208466 z + .07412366556 z^3 + .0007809431668 z^5. \tag{7}$$

Lin [11] gives another sigmoid-like approximation $\Pr(Z > z) \approx \frac{1}{1+\exp(\frac{4.2\pi z}{9-z})}$.

2.2. **Continued fraction expansions.** Expansion

$$\Pr(Z > z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \frac{z}{z + \frac{1}{z + \frac{2}{z + \frac{3}{\cdots}}}}, \tag{8}$$

comes from Laplace [4] and is designed for large $z$, see also [5, 26.2.14 and 26.2.15]. Lee [12] gives an approximation based on optimized truncation of expansion (8). The resulting expressions have a form resembling our 13, and (14), with different constants. Another continued fraction expansion due to Shenton [13] is designed for small $z$.

2.3. **Orthogonal expansion.** Kerridge and Cook [14] compare the performance of several classic expansions to the Hermite polynomial expansion $\Pr(Z > z) = \frac{1}{2} - \sqrt{\frac{2}{\pi}} e^{-z^2/2} \sum_{n=0}^{\infty} (x/2)^{2n+1} \frac{H_{2n}(x/2)}{(2n+1)!}$. Divgi [15] gives the approximation based on expansion of $\Pr(Z > z) e^{z^2/2}$ into orthogonal polynomials with weight function $e^{-x^2}$.

2.4. **Ad hoc approximations.** In this section we list selected "ad hoc approximations"; with the exception of Strecock's ad-hoc approximation, the formulas reviewed below are "pocket calculator" formulas. Waissi & Rossin [1] give several additional references not mentioned here.

2.4.1. *Strecock's approximation.* For $0 \leq z \leq 7$ the most accurate of the "ad hoc' methods is the approximation

$$\Pr(Z > z) \approx \frac{1}{2} - \frac{1}{\pi}\left(\frac{z}{3\sqrt{2}} + \sum_{n=1}^{12} \frac{1}{n} e^{-n^2/9} \sin(nx\sqrt{2}/3)\right) \tag{9}$$

offered by Moran [16] as an improvement on an ingenuous idea of Strecock [17]. The approximation is accurate to nine decimal places for $z \leq 7$, see Section 4, but outside this range the accuracy decreases rapidly. Relative errors increase rapidly for $z > 6$. Notice that although (9) looks like a truncated series expansion, in fact the approximation is constructed with specific accuracy in mind, and thus cannot be readily improved by increasing the number of terms.

2.4.2. *Hart's uniform approximations.* Hart [2] gives a very simple two-constant approximation designed for entire range $0 \leq z < \infty$

$$\Pr(Z > z) \approx \frac{1}{\sqrt{2\pi}} \frac{e^{-z^2/2}}{z + .8e^{-.4z}} \tag{10}$$

This formula has absolute errors of $4.3 \times 10^{-3}$ and uniform relative errors of about 2% over $0 \leq z < \infty$, combining precision and simplicity In another paper, Hart [3] gives a more complex approximation, again designed for all $z \geq 0$ with uniformly small relative error of .055% and absolute errors of $5.3 \times 10^{-5}$.

$$\Pr(Z > z) \approx \frac{e^{-z^2/2}}{\sqrt{2\pi}z}\left(1 - \frac{\sqrt{1 + bz^2}/(1 + az^2)}{(P_0 z + \sqrt{P_0^2 z^2 + \exp(-z^2/2)\sqrt{1 + bz^2}/(1 + az^2)})}\right) \tag{11}$$

where $P_0 = \sqrt{\pi/2}$, $a = \frac{1+\sqrt{1-2\pi^2+6\pi}}{2\pi}$, $b = 2\pi a^2$.

2.4.3. *Other expression.* Several authors combine radicals and exponentials: Johnson and Kotz [19] give $\Pr(Z > z) \approx \frac{1}{2}\sqrt{1 - \exp(-t^2/2)}$. Zelen and Severo [5] offer $\Pr(Z > z) \approx \frac{1}{2} - \sqrt{1 - \exp(-2z^2/\pi)}$; Hamaker [20] extends this approximation to higher order polynomial $\Pr(Z > z) \approx \frac{1}{2} - \sqrt{1 - \exp(-.806z(1 - .018z))}$.

Lin [21] gives a least-square fit $\Pr(Z > z) \approx \frac{1}{2}\exp(-bz - az^2)$ with $a = 0.416, b = 0.717$; Norton [23] gives $a = b = 1/2$, and a slight improvement $\Pr(Z > z) \approx \frac{1}{2}\exp(-1.2z^{0.8})$.

Hastings [22, pg 167] gives a four-constant approximation of the form $\Pr(Z > z) = \frac{1}{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4}$;

## 3. New approximation with small relative errors

3.1. **Rational approximations to Mill's ratio.** For our new uniform approximation, we seek a correction factor in the form of a rational function

$$\Pr(Z > z) \approx \frac{z^n/\sqrt{2\pi} + a_{n-1}z^{n-1} + \cdots + a_0}{z^{n+1} + b_n z^n + \cdots + b_0} e^{-z^2/2} \tag{12}$$

where the two highest order terms are pre-selected to ensure the correct asymptotics as $z \to \infty$, see Feller [18, Section VII.1] and the remaining $2n - 1$ coefficients are determined by matching the derivatives of orders $0, 1, 2 \ldots$ at $z = 0$. Expressions of this form can serve as "pocket calculator" approximations, have accuracy comparable to other similar approximations and do not deteriorate as $z$ increases.

For $n = 1$, formula (12) yields an approximation of satisfactory performance for all $z$

$$\Pr(Z > z) \approx \frac{(4 - \pi)z + \sqrt{2\pi}(\pi - 2)}{(4 - \pi)\sqrt{2\pi}z^2 + 2\pi z + 2\sqrt{2\pi}(\pi - 2)} e^{-z^2/2}.$$

In order to offer a more accurate alternative to (10) we convert this expression into a "calculator approximation" that relies on two numerical constants 3.333 and 7.32 only.

$$\Pr(Z > z) \approx \frac{z + 3.333}{\sqrt{2\pi}z^2 + 7.32z + 2 \times 3.333} e^{-z^2/2} \tag{13}$$

The largest error is .00071 ($7.1 \times 10^{-4}$) occurs in the range $1.07 \leq z \leq 1.13$, and amounts to the third digit being incorrect by at most $\pm 1$. The largest relative error is 2.3% and occurs in the range of $4.8 \leq z \leq 7.8$ Approximation (13) gives one significant digit for the right tail probability for all $0 \leq z < \infty$.

The next level, $n = 2$, expression depends on four constants.

$$\Pr(Z > z) \approx \frac{z^2 + 5.575192695z + 12.77436324}{\sqrt{2\pi}z^3 + 14.38718147z^2 + 31.53531977z + 2 \times 12.77436324} e^{-z^2/2} \tag{14}$$

The largest absolute error is .000019 ($1.9 \times 10^{-5}$) and occurs in the range $1.43 \le z \le 1.61$; it amounts to the fourth digit of the tail probability being incorrect by at most $\pm 1$. The largest relative error is slightly less than .5% and occurs for $z \ge 11.8$. This approximation gives two significant digits for the tail probabilities over all $z > 0$. For example, $\Pr(Z > 11.8) = 1.952 \times 10^{-32} \approx 1.942 \times 10^{-32}$.

## 4. Numerical comparison

We tested all short "pocket calculator" formulas mentioned in the body of the text, and some numerical algorithms. Tables 1 and 2 list the tail probabilities of several more accurate approximations. To facilitate further comparisons with other "pocket calculator" approximations we repeat many of the $z$ values reported in references [20], [10]. Table 3 summarizes the results for "pocket calculator" approximations. As a crude measure of complexity of a formula we chose the number of numerical constants, counting $\pi$ and its simple combinations as half a constant. As measures of precision we list absolute error, and the range where the relative error stays below 50%.

Table 1. Illustration of the accuracy of different approximate formulae for the normal tail probabilities $\Pr(Z > z)$ (small $z$).

| **Approx.** | z=0.1 | z=0.5 | z=1.0 | z=1.5 |
|---|---|---|---|---|
| Exact (2) | .460172162722971 | .308537538725987 | .158655253931457 | .066807201268858 |
| (1)([4]) | .460172162722971 | .3085375387259869 | .15865525393145696 | .06680720126885792 |
| (9)([16]) | .46017216283 | .30853753874 | .15865525391 | .06680720130 |
| (14) | .460172161 | .308536 | .15864 | .06679 |
| (7)([1]) | .46019 | .30856 | .15862 | .06679 |
| (11)([3]) | .4601724 | .30856 | .15871 | .06684 |
| (13) | .4602 | .3088 | .1594 | .0674 |
| (10)([2]) | .45699 | .30482 | .15751 | .06679 |

Table 2. Illustration of the accuracy of different approximate formulae for the normal tail probabilities $\Pr(Z > z)$ (large $z$).

| **Approx.** | z=2.0 | z=3.0 | z=5.0 | z=7.0 | z=9.0 |
|---|---|---|---|---|---|
| Exact (2) | $2.27501319481792 \times 10^{-2}$ | $1.34989803163009 \times 10^{-3}$ | $2.86651571879194 \times 10^{-7}$ | $1.27981254388584 \times 10^{-12}$ | 1.128588 |
| (1)([4]) | $2.2750131948179 \times 10^{-2}$ | $1.3498980316298 \times 10^{-3}$ | $3. \times 10^{-7}$ | (+) | (−) |
| (9)([16]) | $2.27501319 \times 10^{-2}$ | $1.34989796 \times 10^{-3}$ | $2.8654 \times 10^{-7}$ | (−) | (−) |
| (14) | $2.274 \times 10^{-2}$ | $1.348 \times 10^{-3}$ | $2.857 \times 10^{-7}$ | $1.274 \times 10^{-12}$ | $1.123 \times$ |
| (7)([1]) | $2.273 \times 10^{-2}$ | $1.362 \times 10^{-3}$ | $3.732 \times 10^{-7}$ | $64.4 \times 10^{-12}$ | (+) |
| (11)([3]) | $2.276 \times 10^{-2}$ | $1.350 \times 10^{-3}$ | $2.867 \times 10^{-7}$ | $1.27980 \times 10^{-12}$ | 1.12858 |
| (13) | $2.30 \times 10^{-2}$ | $1.37 \times 10^{-3}$ | $2.93 \times 10^{-7}$ | $1.31 \times 10^{-12}$ | $1.15 \times 1$ |
| (10)([2]) | $2.29 \times 10^{-2}$ | $1.37 \times 10^{-3}$ | $2.90 \times 10^{-7}$ | $1.30 \times 10^{-12}$ | $1.14 \times 1$ |

## 5. Conclusions

This paper compares the performance and complexity of several "pocket calculator" formulas for finding standard normal right tail probabilities, see Table 3, and presents two new approximations that have small relative errors. Our approximation (13) has very simple form and can be embedded into other models and formulas that need error of at most $\pm 1$ in the third digit for small $z$ but need to retain one significant digit through the entire range of $z \ge 0$. Our second approximation (14) can be used when error of at most $\pm 1$ in the fourth digit is required for moderate $z$ and two significant digits are required for all $z \ge 0$.

TABLE 3. Accuracy of selected "pocket calculator" approximations, with range where relative error $\frac{|I-A|}{I} < .5$.

| **Citation** | **No of Constants** | **Approximation** | **Absolute Error** | **Rel. Error** $< 50\%$ |
|---|---|---|---|---|
| [10] | 1.5 | (3) & (5) | $1.79 \times 10^{-4}$ | $0 \leq z < 4.13$ |
| [7], [9] | 1.5 | (3) & 2 terms of (4) | $3.1 \times 10^{-4}$ | $0 \leq z < 4.03$ |
| [10] | 2 | (3) & (6) | $1.402 \times 10^{-4}$ | $0 \leq z < 4.18$ |
| — | 2.5 | (13) | $7.1 \times 10^{-4}$ | $0 \leq z < \infty$ |
| [3] | 3 | (11) | $5.32 \times 10^{-5}$ | $0 \leq z < \infty$ |
| [1] | 3 | (3) & (7) | $4.3 \times 10^{-5}$ | $0 \leq z < 5.25$ |
| — | 4.5 | (14) | $1.9 \times 10^{-5}$ | $0 \leq z < \infty$ |
| — | 4.5 | (3) & 5 terms of (4) | $1.7 \times 10^{-5}$ | $0 \leq z < 4.21$ |

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